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Branching rules for classical Lie groups using tensor and spinor methods

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Abstract. Tensor and spinor methods are used to derive branching rule formulae for the embedding of one classical Lie group in another. These formulae involve operations on S-functions. By the judicious use of identities satisfied by certain infinite series of S-functions, they are reduced to forms which may be used very efficiently. Eleven sets of the branching rule formulae derived are as simple as possible, in that they involve only a sum of positive terms, whilst four other sets involve some negative terms which ultimately cancel. The advantage of using a composite notation, both for mixed tensor and for spinor representations, is made apparent. A comparison is made with methods used to derive branching rules based on mapping from one weight space to another.

1. Introduction

Whippman (1965) has published a paper in which he summarized the state of knowledge of the branching rules associated with the embedding of one simple Lie group in another. Now nearly ten years later the time is ripe for an updating of this work. During the intervening period a wealth of different techniques has been developed for tackling the branching rule problem, including those by Itzykson and Nauenberg (1966), Navon and Patera (1967), Hegerfeldt (1967), Delaney and Gruber (1969), Blaha (1969), Wybourne and Butler (1969), Wybourne (1970), Stone (1970), Wong (1970), Gruber (1970), Abramsky and King (1970), King (1971a), and Patera and Sankoff (1972). Some very extensive tables of results have been published by Itzykson and Nauenberg (1966), Wybourne (1970) and Patera and Sankoff (1972).

With one or two exceptions these developments have not followed Whippman in producing results which are independent of the ranks of the particular Lie groups under consideration. Indeed the vast majority of the results obtained have required a distinct calculation for each particular combination of group and subgroup. Whilst this is no embarrassment in the case of exceptional Lie groups, in the case of the families of classical Lie groups this is, to say the least, disappointing. It was these classical Lie groups U(n), O(n) and Sp($2k$) with which Whippman was concerned and which are the concern of the present paper.

The branching rule problem itself is however much more general and may be stated in the following way. Given a group, G, with elements $\{g, \ldots\}$ and irreducible representations $\{\lambda_G, \ldots\}$, and a subgroup, H, with elements $\{h, \ldots\}$ and irreducible representations $\{\mu_H, \ldots\}$, then the restriction of the set of matrices $\{\lambda_G(g)\}$, forming the representation $(\lambda)_G$ of G, to the set $\{\lambda_G(h)\}$ yields a representation of H which is, in general, reducible. If for all h in H

$$\lambda_G(h) = \sum_{\mu_H} m_{\lambda_G}^{\mu_H} \mu_H(h), \tag{1.1}$$

then it is said that under the restriction $G \downarrow H$ the representation $(\lambda)_G$ reduces into a set of irreducible representations any one of which, $(\mu)_H$, occurs with multiplicity $m_{\lambda_G}^{\mu_H}$. The branching rule takes the form:

$$G \downarrow H \qquad (\lambda)_G \downarrow \sum_{\mu_H} m_{\lambda_G}^{\mu_H} (\mu)_H, \tag{1.2}$$

and the problem is the determination of the coefficients $m_{\lambda_G}^{\mu_H}$.

For the subgroup chain $G \downarrow H \downarrow K$ the branching rule appropriate to $G \downarrow K$ is given in terms of those appropriate to $G \downarrow H$ and $H \downarrow K$ by the formula

$$m_{\lambda_G}^{\nu_K} = \sum_{\mu_H} m_{\lambda_G}^{\mu_H} m_{\mu_H}^{\nu_K}, \tag{1.3}$$

where the group K with elements $\{k, \ldots\}$ possesses irreducible representations $\{\nu_K, \ldots\}$.

In a number of important cases the branching rules appropriate to $G \downarrow K$ and $G \downarrow H$ are both known but that appropriate to $H \downarrow K$ is to be determined. If in such a case the matrix with elements $m_{\lambda_G}^{\mu_H}$ possesses a left inverse with elements $b_{\mu_H}^{\lambda_G}$, then (1.1) may be inverted and the restriction of the group elements under consideration to be elements of K then gives:

$$\mu_H(k) = \sum_{\lambda_G} b_{\mu_H}^{\lambda_G} \lambda_G(k). \tag{1.4}$$

Application of the formula (1.1) to $G \downarrow K$ rather than $G \downarrow H$ then yields for the branching rule coefficients appropriate to $H \downarrow K$ the formula

$$m_{\mu_H}^{\nu_K} = \sum_{\lambda_G} b_{\mu_H}^{\lambda_G} m_{\lambda_G}^{\nu_K}. \tag{1.5}$$

Denoting the inverse of the restriction $G \downarrow H$ by $H \uparrow_{\mathrm{r}} G$ this result may be written in the following way:

$$H \uparrow_{\mathrm{r}} G \downarrow K \qquad \mu_H \uparrow_{\mathrm{r}} \sum_{\lambda_G} b_{\mu_H}^{\lambda_G} \lambda_G \downarrow \sum_{\lambda_G, \nu_K} b_{\mu_H}^{\lambda_G} m_{\lambda_G}^{\nu_K} \nu_K. \tag{1.6}$$

The subscript r in the symbol $\uparrow_{\mathrm{r}}$ has been introduced to avoid confusion with the symbol $\uparrow$ conventionally used to signify induction. Furthermore the matrix with elements $m_{\lambda_G}^{\mu_H}$ may not possess any left inverse, and if it does so this inverse may not be unique. The subscript r serves as a reminder of a possible lack of uniqueness. It has been shown for finite groups G and H that the necessary and sufficient condition for the existence of an inverse matrix with elements $b_{\mu_H}^{\lambda_G}$ is simply that the restriction $G \downarrow H$ should involve no splitting of classes (Robinson 1973, Robinson 1975, Backhouse 1974, private communication). The proof of this result may be extended to cover the case of the compact groups considered in this paper. Indeed in many cases, including those considered here, the existence of an inverse matrix with elements $b_{\mu_H}^{\lambda_G}$ may be seen by an inspection of the set of coefficients $m_{\lambda_G}^{\mu_H}$. Any lack of uniqueness of the operation $\uparrow_{\mathrm{r}}$ in $H \uparrow_{\mathrm{r}} G$ is irrelevant in (1.6) since the subsequent application of the restriction operation $\downarrow$ in $G \downarrow K$ is a restriction to a subgroup K of H.

It should be stressed that in general a subgroup H of G may be embedded in G in more than one distinct way. The branching rules associated with distinct embeddings are themselves distinct. An examination of the nature of each embedding is crucial in dealing with chains of groups and their subgroups. In all cases embeddings may be defined in terms of the branching rules appropriate to just one or two irreducible representations.

In the next section the notation is established for partitions and for S-functions, which are used in the specification of irreducible representations of U(n), O(n) and Sp($2k$). This notation is very close to that used by Whippman (1965), who followed that of Murnaghan (1938), but some alterations and generalizations are adopted in order to make the connection with tensor and spinor realizations of these representations transparent. In particular the composite notation (Abramsky 1969, King 1970a) appropriate to mixed tensors is used, as well as a similar composite notation appropriate to spinors. This section closes with a statement of the modification rules leading to equivalence relations between various irreducible representations and a list of all inequivalent irreducible representations of U(n), O(n) and Sp($2k$), together with a set of dimensionality formulae.

The duality between the symmetric group and the unitary group which arises directly from the existence of tensor representations is made precise through the use of S-functions. The fundamental operations on S-functions are defined in § 3, in which some important series of S-functions introduced by Littlewood (1950a) are defined and relationships between them given.

Almost all of the embeddings considered by Whippman are discussed in § 4, in which a complete set of branching rules is established for all irreducible covariant tensor representations. These rules are extended to cover mixed tensor and spinor representations in § 5. The remaining embedding considered by Whippman involves inner products of S-functions in the statement of its branching rule. This rule is written down in § 6 along with some generalizations.

In the concluding section the results obtained are discussed within the context of the problem of the determination of all branching rules appropriate to the embedding of one Lie group in another. Comparison is made with the complementary work of Patera and Sankoff (1972).

2. Irreducible representations of the classical Lie groups

Before discussing the classical Lie groups it is convenient to introduce the notation for partitions and S-functions. A partition of l into p parts $\lambda_1, \lambda_2, \ldots, \lambda_p$, with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p > 0$, is denoted by $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ or more shortly by (λ). As a general rule corresponding Greek and Latin letters, such as λ and l, are used to specify a partition and its weight, although a partition into one part only is denoted by a Latin letter, so that (λ_1) is written as (l). The partition conjugate to (λ) is denoted by $(\tilde{\lambda})$.

Partitions may be used to label the irreducible representations of the symmetric groups. Corresponding to each irreducible representation there exists a particular symmetric function of a set of indeterminates known as an S-function (Littlewood 1950a, p84). Such an S-function is then labelled by a partition and is written as $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$, or merely as $\{\lambda\}$ if the corresponding parts need not be displayed.

The connection between S-functions and the classical Lie groups comes about through the fact that an S-function $\{\lambda\}$ of the characteristic roots of a matrix A is the trace of an invariant matrix, $A^{\{\lambda\}}$, whose elements are polynomials in the elements of A. If the matrix A is an element of a group the map onto $A^{\{\lambda\}}$ gives a representation of that group whose character is the S-function $\{\lambda\}$. The direct product of the group characters is isomorphic with the outer product of S-functions. The invariant matrix $A^{\{\lambda\}}$ is a particular symmetrized tensor power of A. The indices of the tensors forming the basis of the representation of the group realized by the matrices $A^{\{\lambda\}}$ themselves form the basis of the irreducible representation of the symmetric group S_l labelled by the partition $\{\lambda\}$.

In the case of the group U(n) the representation of the $n\times n$ matrices A by the matrices $A^{\{\lambda\}}$ is irreducible and is conventionally denoted by $\{\lambda\}$. For the groups O(n) and Sp($2k$) further reduction is possible by virtue of the existence of symmetric and antisymmetric metric tensors. The irreducible representations of O(n) and Sp($2k$) are denoted by $[\lambda]$ and $\langle\lambda\rangle$ respectively, where the distinguishing types of bracket are used to signify the tracelessness of the basis tensors under contractions with the appropriate metric tensor.

For the group U(n) not all irreducible representations may be labelled by $\{\lambda\}$ as was pointed out for example by Weyl (1931, p381). There exist inequivalent irreducible representations associated with mixed tensors. These are denoted by $\{\bar{\mu};\lambda\}$ where $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ and $(\mu) = (\mu_1, \mu_2, \ldots, \mu_q)$ are partitions of l and m into p and q non-vanishing parts. This composite notation (Abramsky 1969, King 1970a, Abramsky and King 1970) implies that the mixed tensors which afford a realization of the representation $\{\bar{\mu};\lambda\}$ are traceless under contractions between covariant and contravariant indices, whose symmetry properties are defined by the partitions (λ) and (μ) respectively. The bar in the composite symbol, over μ, emphasizes the contravariant nature of the associated indices. In the absence of such indices $(\mu) = (0)$ and $\{\bar{0};\lambda\} = \{\lambda\}$, whilst in the absence of covariant indices $(\lambda) = (0)$ and $\{\bar{\mu};0\} = \{\bar{\mu}\}$.

The groups O(n) possess double-valued representations known as spinor representations (Brauer and Weyl 1935) in addition to the true, single-valued tensor representations. It is convenient to denote such spinor representations by $[\Delta;\lambda]$ where Δ signifies the presence of a spinor index in the basis states, whilst (λ) is a partition giving the symmetry properties of the tensor indices of the basis states. The composite notation this time implies that multiplication by matrices which are elements of the associated Clifford algebra and contraction of the index of such a matrix with a tensor index gives zero. This condition is necessary for irreducibility.

Taken together with the covariant tensor representations $\{\lambda\}$, $[\lambda]$ and $\langle\lambda\rangle$, these two types of composite representations $\{\bar{\mu};\lambda\}$ and $[\Delta;\lambda]$ complete the list of all irreducible representations of U(n), O(n) and Sp($2k$). Among these representations there exists for each group a one-dimensional irreducible representation, denoted in all cases by ϵ, in which each group element A is mapped onto its determinant: det A. In the case of Sp($2k$) all the group elements are unimodular so that ϵ is the trivial, identity representation $\langle 0\rangle$. However in the case of U(n), ϵ is the representation $\{1^n\}$ whose basis is a totally antisymmetric nth rank covariant tensor. Its contragredient $\bar{\epsilon}$ is the representation $\{\overline{1^n}\}$ whose basis is a totally antisymmetric nth rank contravariant tensor, and for which each group element A is mapped onto $(\det A)^{-1}$, so that it is natural to write $\bar{\epsilon} = \epsilon^{-1}$. For O($n$), ϵ is the representation $[1^n]$, but in this case $\det A = \pm 1$ for all group elements, so that ϵ and $\bar{\epsilon}$ coincide.

The product of any irreducible representation with any power of ϵ is necessarily irreducible. Hence certain inequivalent irreducible representations are related through powers of ϵ. They may be said to be associated.

For any irreducible $\{\bar{\mu};\lambda\}$ of U(n), with a mixed tensor basis, there always exists an associated irreducible representation $\{\rho\}$, with a covariant basis, such that:

$$\mathrm{U}(n) \qquad \{\bar{\mu};\lambda\} = \epsilon^{w}\{\rho\} \tag{2.1}$$

for some integer w, as implied by Weyl (1931, p381). The relationship between w and

(ρ) and (λ) and (μ) has been given elsewhere (King 1970a, 1971a). Unfortunately this relationship is necessarily n-dependent in that for given (λ) and (μ) both w and (ρ) will be different for different values of n.

In the case of O(n) it is clear that $\epsilon^2 = [0]$ since $\det A = \pm 1$ for all group elements A. Hence there is at most one inequivalent associate of any irreducible representation. Moreover a representation and its associate, although always distinct in principle, will be equivalent and thus self-associate, if the character of the representation vanishes for all group elements satisfying $\det A = -1$. In general it is conventional following Murnaghan (1938, p276) to write:

$$\mathrm{O}(n) \qquad [\lambda]^* = \epsilon[\lambda] \qquad [\Delta;\lambda]^* = \epsilon[\Delta;\lambda]. \tag{2.2}$$

All spinor representations of O$(2k)$ are self-associate, as well as those tensor representations $[\lambda]$ for which (λ) is a partition into exactly k non-vanishing parts. On the other hand no irreducible representation of O$(2k+1)$ is self-associate. Unfortunately the spinor representations of O$(2k+1)$ are necessarily constructed by viewing this group as the direct product SO$(2k+1)\times Z_2$ where the elements of Z_2 are ± 1. This means that the double-valued ambiguity in the spin representations of SO$(2k+1)$ is compounded when the extension is made to O$(2k+1)$ (Brauer and Weyl 1935). It follows that $[\Delta;\lambda]$ and $[\Delta;\lambda]^*$, although logically distinct for O$(2k+1)$, are each ambiguous up to a factor of ± 1, which is the factor which supposedly distinguishes them.

Quite apart from the use of ϵ and $\bar{\epsilon}$ to obtain relationships between inequivalent irreducible representations, the corresponding antisymmetric tensors may be used to derive equivalence relationships and the corresponding modification rules. These rules are best expressed in terms of a hook removal procedure applied to the Young diagrams corresponding to the representations under consideration. The list of rules given earlier (King 1971a) may be extended slightly to yield:

$$\begin{array}{lll} \mathrm{U}(n) & \{\bar{\mu};\lambda\} = (-1)^{x+y-1}\{\overline{\mu-h};\lambda-h\} & \text{with } h = p+q-n-1 \\ \mathrm{O}(n) & [\lambda] = (-1)^{x-1}[\lambda-h]^* & \text{with } h = 2p-n \\ & [\Delta;\lambda] = (-1)^{x}[\Delta;\lambda-h]^* & \text{with } h = 2p-n-1 \\ \mathrm{Sp}(2k) & \langle\lambda\rangle = (-1)^{x}\langle\lambda-h\rangle & \text{with } h = 2p-2k-2 \end{array} \tag{2.3}$$

where h is the length of the continuous boundary hook removed from the Young diagrams corresponding to (λ) and (μ), starting from the foot of the first column and ending in the xth and yth column respectively. The corresponding representation vanishes identically unless after the hook removal a regular Young diagram, corresponding to some new partition, is obtained. The symbols $(\lambda-h)$ and $(\mu-h)$ specify such new partitions. For example if $(\lambda) = (43^22^3)$ and $h = 4$ then $(\lambda-h) = (43^21^2)$ whilst if $(\lambda) = (43^32^2)$ and $h = 4$ then $(\lambda-h) = 0$. Continued application of these rules, or the use of other techniques due to Murnaghan (1938, p 282, 1958 p 121, 1962 p 129) and Newell (1951), leads to the existence of the following list of all inequivalent irreducible representations of the groups under consideration:

$$\begin{array}{lll} \mathrm{U}(k) & \{\bar{\mu};\lambda\} & \text{with } p+q \leqslant k \\ \mathrm{O}(2k) & [\lambda], [\lambda]^* & \text{with } p < k \\ & [\lambda] & \text{with } p = k \\ & [\Delta;\lambda] & \text{with } p \leqslant k \end{array}$$

$$
\begin{array}{lll}
O(2k+1) & [\lambda], [\lambda]^* & \text{with } p \leqslant k \\
 & [\Delta;\lambda], [\Delta;\lambda]^* & \text{with } p \leqslant k \\
Sp(2k) & \langle\lambda\rangle & \text{with } p \leqslant k,
\end{array}
\tag{2.4}
$$

where (λ) and (μ) are arbitrary partitions consisting of p and q non-vanishing parts.

The double-valued representations of O(n) owe their origin to the fact that the universal covering group of the Lie algebra associated with O(n) is the group Spin (n). Similarly there exist multi-valued representations of U(n) owing their origin to the fact that the corresponding covering group is SU(n) $\times$ R where R is the additive group of real numbers. If A is any element of U(n) then det $A = e^{i\phi}$ with ϕ real. The mapping of A onto $e^{ir\phi}$ gives a representation ϵ^r of U(n) which is multi-valued if r is not integral. Further inequivalent irreducible multi-valued representations of U(n) then take the form $\epsilon^r\{\bar{\mu};\lambda\}$ for non-integral values of r. These are not normally considered but have a role to play in one of the branching rules considered in § 5.

Various formulae have been given which enable the dimensions of the irreducible representations of the classical Lie groups to be calculated. Those that are most appropriate for dealing with these families of groups, rather than just individual groups, take the form:

$$
\begin{array}{ll}
U(k) & D_k\{\bar{\mu};\lambda\} = N_k\{\bar{\mu};\lambda\}/H(\lambda)H(\mu) \\
O(2k) & D_{2k}[\lambda] = N_{2k}[\lambda]/H(\lambda) \\
 & D_{2k}[\Delta;\lambda] = 2^k N_{2k-1}\langle\lambda\rangle/H(\lambda) \\
O(2k+1) & D_{2k+1}[\lambda] = N_{2k+1}[\lambda]/H(\lambda) \\
 & D_{2k+1}[\Delta;\lambda] = 2^k N_{2k}\langle\lambda\rangle/H(\lambda) \\
Sp(2k) & D_{2k}\langle\lambda\rangle = N_{2k}\langle\lambda\rangle/H(\lambda)
\end{array}
\tag{2.5}
$$

where use has been made of the link between symplectic group characters and those of spinor representations of the orthogonal group, pointed out by Bauer (1954). The denominator functions $H(\lambda)$ and $H(\mu)$ are the products of the hook length factors, introduced by Robinson (1961, p 44), associated with the Young diagrams specified by (λ) and (μ). Diagrammatic methods have been given which enable $N_n\{\bar{\mu};\lambda\}$, $N_n[\lambda]$ and $N_n\langle\lambda\rangle$ to be written down as factored polynomials in n (King 1970b, 1971b, Abramsky *et al* 1973). These formulae may be used to check the validity of the modification rules in any particular case under consideration. They are also of use in checking all applications of the branching rule formulae given in the following sections of this paper.

3. *S*-function series

Whereas partitions are merely labelling devices, *S*-functions are subject to a number of distinct operations including addition, subtraction, outer multiplication, inner multiplication and division. These operations are denoted by $+$, $-$, $.$, $\circ$ and $/$ respectively. They all correspond to operations on representations of U(n) and on the tensor bases of such representations.

Addition is trivial and corresponds to the direct summation of the matrices of representations. Subtraction is the inverse of this operation. Outer multiplication of *S*-functions corresponds to the tensor multiplication of representations of U(n) and to the mutual symmetrization of sets of tensor indices. The coefficients $m^{\mu}_{\sigma\tau}$ appearing in the reduction of the Kronecker or tensor product

$$\mathrm{U}(n) \qquad \{\sigma\}\,.\,\{\tau\} = \{\sigma\,.\,\tau\} = \sum_{\mu} m^{\mu}_{\sigma\tau}\{\mu\}, \tag{3.1}$$

may be evaluated by means of the Littlewood–Richardson rule (Littlewood 1950a, p 94). The same coefficients appear in the evaluation of the quotient (Littlewood 1950a, p 110):

$$\{\mu\}/\{\sigma\} = \{\mu/\sigma\} = \sum_{\tau} m^{\mu}_{\sigma\tau}\{\tau\}. \tag{3.2}$$

The relationship between outer products and quotients is such that

$$\sum_{\sigma}(-1)^{s}\{\mu/\sigma\}\,.\,\{\tilde{\sigma}\} = \delta_{\mu 0}\{0\} \tag{3.3}$$

where $\{0\}$ is the trivial S-function, ie $\{0\} = 1$.

The division of S-functions corresponds to carrying out contractions between sets of tensor indices. It is this fact which has led to the determination of very succinct formulae for the reduction of Kronecker products of representations of the classical groups (Littlewood 1958, Abramsky and King 1970, King 1971a). Amongst these is the formula corresponding to the systematic extraction of traces in a product of covariant and contravariant tensors:

$$\mathrm{U}(n) \qquad \{\lambda\}\,.\,\{\bar{\mu}\} = \sum_{\zeta}\{\overline{\mu/\zeta}\,;\lambda/\zeta\}. \tag{3.4}$$

By virtue of (3.3) this expression has as its inverse

$$\mathrm{U}(n) \qquad \{\bar{\mu};\lambda\} = \sum_{\zeta}(-1)^{z}\{\overline{\mu/\zeta}\}\,.\,\{\lambda/\zeta\}. \tag{3.5}$$

The inner multiplication of S-functions corresponds to the Kronecker multiplication of representations of the symmetric group, and also to the overall symmetrization of sets of tensor indices with respect to simultaneous permutations of members of those sets. The coefficients $k_{\sigma\tau\mu}$ appearing in the reduction of an inner product

$$\{\sigma\}\circ\{\tau\} = \{\sigma\circ\tau\} = \sum_{\mu} k_{\sigma\tau\mu}\{\mu\} \tag{3.6}$$

may be evaluated in many ways, the most efficient of which is probably that given by Robinson (1961, p 64). In (3.6) it is to be noted that (σ), (τ) and (μ) are necessarily partitions of the same number, ie $s = t = m$. The coefficients $k_{\sigma\tau\mu}$ are totally symmetric under permutations of σ, τ, and μ so that no independent operation of 'inner division' exists; it corresponds exactly to inner multiplication.

It was Littlewood (1950a, p 238) who pointed out the importance of certain infinite series of S-functions. The list of such series can usefully be extended to include the following:

$$A = \sum_{\alpha}(-1)^{a/2}\{\alpha\} \qquad B = \sum_{\beta}\{\beta\}$$

$$C = \sum_{\gamma}(-1)^{c/2}\{\gamma\} \qquad D = \sum_{\delta}\{\delta\}$$

$$E = \sum_{\varepsilon}(-1)^{(e+r)/2}\{\varepsilon\} \qquad F = \sum_{\zeta}\{\zeta\}$$

$$G = \sum_{\varepsilon}(-1)^{(e-r)/2}\{\varepsilon\} \qquad H = \sum_{\zeta}(-1)^{z}\{\zeta\}$$

$$L = \sum_m (-1)^m \{1^m\} \qquad M = \sum_m \{m\}$$

$$P = \sum_m (-1)^m \{m\} \qquad Q = \sum_m \{1^m\}$$

where (m) is a partition into one part only, (1^m) is the conjugate of (m), (δ) is a partition each of whose parts is even, (β) is the conjugate of (δ), (ζ) is any partition, (ε) is any self-conjugate partition of rank r so that in Frobenius notation it takes the form:

$$(\varepsilon) = \begin{pmatrix} e_1 & e_2 & \dots & e_r \\ e_1 & e_2 & \dots & e_r \end{pmatrix}$$

whilst (γ) and (α) are mutually conjugate partitions which in Frobenius notation take the form:

$$(\gamma) = \begin{pmatrix} c_1+1 & c_2+1 & \dots & c_r+1 \\ c_1 & c_2 & \dots & c_r \end{pmatrix}, \qquad (\alpha) = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ a_1+1 & a_2+1 & \dots & a_r+1 \end{pmatrix}.$$

It is not difficult to see that these series have been placed in mutually inverse pairs in that

$$AB = CD = EF = GH = LM = PQ = \{0\} = 1, \tag{3.7}$$

where juxtaposition of S-function series indicates the formation of outer products of S-functions, one taken from each series, and the summation of all such products. Furthermore:

$$\begin{aligned} LA = PC = E, \qquad & MB = QD = F, \\ MC = QA = G, \qquad & LD = PB = H, \end{aligned} \tag{3.8}$$

whilst

$$BD = \sum_\xi \{\xi\} . \{\xi\}, \tag{3.9}$$

and, more generally, for any S-function $\{\zeta\}$

$$BD . \{\zeta\} = \sum_\xi \{\xi\} . \{\xi/\zeta\}. \tag{3.10}$$

As a preliminary application of these infinite series it is worth considering the reduction of the Kronecker product of the fundamental spin representation and a tensor representation of O(n). This reduction corresponds to the contraction of tensor indices with antisymmetric products of elements of the Clifford algebra. The appropriate formula for O($2k$) and O($2k+1$) is then

$$\Delta . [\lambda] = \sum_{m=0}^{k} [\Delta ; \lambda/1^m]. \tag{3.11}$$

Provided that the modification rules are used to ensure that this formula is used only in cases for which (λ) is a partition consisting of p non-vanishing parts with $p \leqslant k$, the summation may be extended indefinitely to give

$$\mathrm{O}(n) \qquad \Delta . [\lambda] = \sum_m [\Delta ; \lambda/1^m] = [\Delta ; \lambda/Q]. \tag{3.12}$$

The fact that Q has an inverse P then implies that

$$\mathrm{O}(n) \qquad [\Delta;\lambda] = \sum_m (-1)^m \Delta\,.\,[\lambda/m] = \Delta\,.\,[\lambda/P]. \tag{3.13}$$

Although both Murnaghan (1938, p. 313) and Littlewood (1950b, p 283) have given the result (3.12), its inverse has not appeared previously and neither result seems to be sufficiently well known. In what follows they are used to derive branching rules.

4. Branching rules for covariant tensor representations

The group U(n) contains four very important subgroups U($n-1$), O(n), Sp($2k$) if $n = 2k$ and U(p)×U(q) if $n = p+q$. If the elements of U(n) are the $n \times n$ unitary matrices A, then these subgroups are those arising from restricting A to be such that $A = B + 1$, $A^{\mathrm{T}}A = I$, $A^{\mathrm{T}}JA = J$ and $A = C + D$ where B, C and D are unitary matrices of dimension $(n-1)$, p and q respectively, whilst I and J are the symmetric and antisymmetric matrices whose elements are the metric tensors of O(n) and Sp($2k$).

These restrictions correspond to the reductions of the defining representation, $\{1\}$, of U(n) given by:

$$\mathrm{U}(n)\downarrow \mathrm{U}(n-1) \qquad \{1\}\downarrow\{1\}+\{0\} \tag{4.1}$$

$$\mathrm{U}(n)\downarrow \mathrm{O}(n) \qquad \{1\}\downarrow[1] \tag{4.2}$$

$$\mathrm{U}(2k)\downarrow \mathrm{Sp}(2k) \qquad \{1\}\downarrow\langle 1\rangle \tag{4.3}$$

$$\mathrm{U}(p+q)\downarrow \mathrm{U}(p)\times\mathrm{U}(q) \qquad \{1\}\downarrow\{1\}\times\{0\}+\{0\}\times\{1\}. \tag{4.4}$$

The corresponding branching rules for the covariant tensor representations of U(n) are:

$$\mathrm{U}(n)\downarrow \mathrm{U}(n-1) \qquad \{\lambda\}\downarrow\sum_m\{\lambda/m\} = \{\lambda/M\} \tag{4.5}$$

$$\mathrm{U}(n)\downarrow \mathrm{O}(n) \qquad \{\lambda\}\downarrow\sum_\delta[\lambda/\delta] = [\lambda/D] \tag{4.6}$$

$$\mathrm{U}(2k)\downarrow \mathrm{Sp}(2k) \qquad \{\lambda\}\downarrow\sum_\beta\langle\lambda/\beta\rangle = \langle\lambda/B\rangle \tag{4.7}$$

$$\mathrm{U}(p+q)\downarrow \mathrm{U}(p)\times\mathrm{U}(q) \qquad \{\lambda\}\downarrow\sum_\zeta\{\lambda/\zeta\}\times\{\zeta\}, \tag{4.8}$$

where the notation introduced in § 3 for infinite series of S-functions has been used to simplify the statement of the first three rules.

The rules may be justified using the following tensorial arguments:

U(n) ↓ U($n-1$). The spaces corresponding to tensors for which a particular number of indices, say m, take on the value n, define invariant subspaces. Such indices must be mutually symmetrized. The irreducible representations specified by the quotient $\{\lambda/m\}$ are those corresponding to tensors obtained by contracting the indices of the tensor corresponding to $\{\lambda\}$ with an mth rank symmetric tensor. The symbol M denotes the sum of all totally symmetric S-functions. The rule (4.5) was first given by Weyl (1931, p. 391).

U(n) ↓ O(n). Invariant subspaces are formed through successive contractions with the symmetric second rank metric tensor. The symmetric product of such metric tensors contains terms corresponding to partitions all of whose parts are even. Such partitions are denoted by (δ) and the sum of all such S-functions is denoted by D. This statement (4.6) of the branching rule is due to Littlewood (1950a, p 240).

$\mathrm{U}(2k) \downarrow \mathrm{Sp}(2k)$. In this case the metric tensor is antisymmetric, the appropriate partitions (β) are the conjugates of (δ), and the corresponding S-function series is B. This rule is also due to Littlewood (1950a, p 295).

$\mathrm{U}(p+q) \downarrow \mathrm{U}(p) \times \mathrm{U}(q)$. For this restriction the set of tensor indices associated with $\{\lambda\}$ divides into two sets. The symmetries of the indices taking on the values $1, 2, \ldots, p$ and those taking on the values $p+1, p+2, \ldots, p+q$, given by partitions (η) and (ζ) respectively, correspond to a tensor whose overall symmetry is given by (λ) provided that the outer product, $\{\eta\} . \{\zeta\}$, contains $\{\lambda\}$ or equivalently that $\{\eta\}$ is one of the terms in the S-function quotient $\{\lambda/\zeta\}$.

Whippman (1965) gave the rule (4.8) in precisely the form given here, but expressed the rules (4.5), (4.6) and (4.7) rather differently. The S-function notation of Littlewood (1950a) adopted here not only has the advantage of showing the tensor origin of the reduction process but also lends itself very well to the inversion of the restriction procedure. In fact from the S-function formula (3.7):

$$M^{-1} = L, \qquad D^{-1} = C, \qquad B^{-1} = A.$$

Using the symbol $\uparrow_{\mathrm{r}}$ to indicate the inverse of the restriction process, as discussed in the introduction, it then follows that the rules (4.5), (4.6) and (4.7) may be inverted to give:

$$\mathrm{U}(n-1) \uparrow_{\mathrm{r}} \mathrm{U}(n) \qquad \{\lambda\} \uparrow_{\mathrm{r}} \{\lambda/L\} \tag{4.9}$$

$$\mathrm{O}(n) \uparrow_{\mathrm{r}} \mathrm{U}(n) \qquad [\lambda] \uparrow_{\mathrm{r}} \{\lambda/C\} \tag{4.10}$$

$$\mathrm{Sp}(2k) \uparrow_{\mathrm{r}} \mathrm{U}(2k) \qquad \langle\lambda\rangle \uparrow_{\mathrm{r}} \{\lambda/A\}. \tag{4.11}$$

It is then a simple matter to derive the branching rules appropriate to $\mathrm{O}(n) \downarrow \mathrm{O}(n-1)$ and $\mathrm{Sp}(2k) \downarrow \mathrm{Sp}(2k-2)$ through the application of (1.6) to the chains:

$$\mathrm{O}(n) \uparrow_{\mathrm{r}} \mathrm{U}(n) \downarrow \mathrm{U}(n-1) \downarrow \mathrm{O}(n-1)$$

$$\mathrm{Sp}(2k) \uparrow_{\mathrm{r}} \mathrm{U}(2k) \downarrow \mathrm{U}(2k-1) \downarrow \mathrm{U}(2k-2) \downarrow \mathrm{Sp}(2k-2).$$

This yields

$$[\lambda] \uparrow_{\mathrm{r}} \{\lambda/C\} \downarrow \{\lambda/CM\} \downarrow [\lambda/CMD]$$

$$\langle\lambda\rangle \uparrow_{\mathrm{r}} \{\lambda/A\} \downarrow \{\lambda/AM\} \downarrow \{\lambda/AMM\} \downarrow \langle\lambda/AMMB\rangle.$$

The use of (3.7) then gives the branching rules:

$$\mathrm{O}(n) \downarrow \mathrm{O}(n-1) \qquad [\lambda] \downarrow \sum_{m} [\lambda/m] = [\lambda/M] \tag{4.12}$$

$$\mathrm{Sp}(2k) \downarrow \mathrm{Sp}(2k-2) \qquad \langle\lambda\rangle \downarrow \sum_{m,m'} \langle\lambda/m . m'\rangle = \langle\lambda/MM\rangle. \tag{4.13}$$

The first of these is due originally to Boerner (1970, p 267), and was quoted by Whippman, whilst the second is due to Miller (1966) and Hegerfeldt (1967).

In exactly the same way an analysis of the chains

$$\mathrm{O}(p+q)\uparrow_r \mathrm{U}(p+q)\downarrow \mathrm{U}(p)\times\mathrm{U}(q)\downarrow \mathrm{O}(p)\times\mathrm{O}(q)$$

$$\mathrm{Sp}(2p+2q)\uparrow_r \mathrm{U}(2p+2q)\downarrow \mathrm{U}(2p)\times\mathrm{U}(2q)\downarrow \mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$$

gives:

$$[\lambda]\uparrow_r \{\lambda/C\}\downarrow \sum_{\zeta}\{\lambda/C\zeta\}\times\{\zeta\}\downarrow \sum_{\zeta}[\lambda/C\zeta D]\times[\zeta/D]$$

$$\langle\lambda\rangle\uparrow_r \{\lambda/A\}\downarrow \sum_{\zeta}\{\lambda/A\zeta\}\times\{\zeta\}\downarrow \sum_{\zeta}\langle\lambda/A\zeta B\rangle\times\langle\zeta/B\rangle.$$

It follows from (3.7) that the branching rules are then

$$\mathrm{O}(p+q)\downarrow \mathrm{O}(p)\times\mathrm{O}(q) \qquad [\lambda]\downarrow \sum_{\zeta,\delta}[\lambda/\zeta]\times[\zeta/\delta] = \sum_{\zeta}[\lambda/\zeta]\times[\zeta/D] \tag{4.14}$$

$$\mathrm{Sp}(2p+2q)\downarrow \mathrm{Sp}(2p)\times\mathrm{Sp}(2q) \qquad \langle\lambda\rangle\downarrow \sum_{\zeta,\beta}\langle\lambda/\zeta\rangle\times\langle\zeta/\beta\rangle = \sum_{\zeta}\langle\lambda/\zeta\rangle\times\langle\zeta/B\rangle. \tag{4.15}$$

These new results, which are a direct generalization of (4.12) and (4.13), in just the same way that (4.8) is a generalization of (4.5), not only take a very simple form involving a direct sum of positive terms but also have a simple interpretation. The tensors associated with $[\lambda/\zeta]$ and $\langle\lambda/\zeta\rangle$ are traceless, since those associated with $[\lambda]$ and $\langle\lambda\rangle$ are themselves traceless. However the divisor in a quotient of S-functions corresponding to contractions does not necessarily correspond to a traceless tensor. The branching rules (4.14) and (4.15) are then obtained using the argument leading to (4.8) for the unitary group, together with the application of the reductions (4.6) and (4.7) to the tensors associated with $\{\zeta\}$.

Three branching rules considered by Whippman, and not yet discussed here, are those appropriate to the restrictions $\mathrm{O}(2k+1)\downarrow\mathrm{U}(k)$, $\mathrm{O}(2k)\downarrow\mathrm{U}(k)$ and $\mathrm{Sp}(2k)\downarrow\mathrm{U}(k)$. These rules can be determined from an analysis of the chains

$$\mathrm{O}(2k+1)\downarrow \mathrm{O}(2k)\uparrow_r \mathrm{U}(2k)\downarrow \mathrm{U}(k)\times\mathrm{U}(k)\downarrow \mathrm{U}(k)$$

$$\mathrm{Sp}(2k)\uparrow_r \mathrm{U}(2k)\downarrow \mathrm{U}(k)\times\mathrm{U}(k)\downarrow \mathrm{U}(k).$$

The particular embedding of $\mathrm{U}(k)\times\mathrm{U}(k)$ in $\mathrm{U}(2k)$ relevant here is not defined as a special case of (4.4) but rather by

$$\mathrm{U}(2k)\downarrow \mathrm{U}(k)\times\mathrm{U}(k) \qquad \{1\}\downarrow\{1\}\times\{0\}+\{0\}\times\{\bar{1}\}, \tag{4.16}$$

and the corresponding branching rule is

$$\mathrm{U}(2k)\downarrow \mathrm{U}(k)\times\mathrm{U}(k) \qquad \{\lambda\}\downarrow\sum_{\xi}\{\lambda/\xi\}\times\{\bar{\xi}\}. \tag{4.17}$$

The final link in the chains corresponds to identifying elements within the two groups $\mathrm{U}(k)$ of $\mathrm{U}(k)\times\mathrm{U}(k)$ and forming the Kronecker product of the representation matrices. From (3.4) it follows that

$$\mathrm{U}(k)\times\mathrm{U}(k)\downarrow \mathrm{U}(k) \qquad \{\eta\}\times\{\bar{\xi}\}\downarrow\sum_{\rho}\{\overline{\xi/\rho};\eta/\rho\}. \tag{4.18}$$

Combining (4.17) and (4.18) and changing the summation variables leads to the rule

$$\mathrm{U}(2k)\downarrow \mathrm{U}(k) \qquad \{\lambda\}\downarrow\sum_{\xi,\zeta}\{\bar{\zeta};\lambda/\xi\cdot(\xi/\zeta)\} = \sum_{\zeta}\{\bar{\zeta};\lambda/BD\zeta\}, \tag{4.19}$$

where use has been made of the remarkable identity (3.10).

The application of (4.12), (4.10) and (4.11) then gives

$$\mathrm{O}(2k+1) \downarrow \mathrm{U}(k) \qquad [\lambda] \downarrow \sum_{\zeta} \{\bar{\zeta}; \lambda/MCBD\zeta\}$$

$$\mathrm{O}(2k) \downarrow \mathrm{U}(k) \qquad [\lambda] \downarrow \sum_{\zeta} \{\bar{\zeta}; \lambda/CBD\zeta\}$$

$$\mathrm{Sp}(2k) \downarrow \mathrm{U}(k) \qquad \langle\lambda\rangle \downarrow \sum_{\zeta} \{\bar{\zeta}; \lambda/ABD\zeta\}.$$

Using (3.7) yet again, it follows that

$$\mathrm{O}(2k+1) \downarrow \mathrm{U}(k) \qquad [\lambda] \downarrow \sum_{\zeta,\beta,m} \{\bar{\zeta}; \lambda/\zeta\beta m\} = \sum_{\zeta} \{\bar{\zeta}; \lambda/BM\zeta\} \tag{4.20}$$

$$\mathrm{O}(2k) \downarrow \mathrm{U}(k) \qquad [\lambda] \downarrow \sum_{\zeta,\beta} \{\bar{\zeta}; \lambda/\zeta\beta\} = \sum_{\zeta} \{\bar{\zeta}; \lambda/B\zeta\} \tag{4.21}$$

$$\mathrm{Sp}(2k) \downarrow \mathrm{U}(k) \qquad \langle\lambda\rangle \downarrow \sum_{\zeta,\delta} \{\bar{\zeta}; \lambda/\zeta\delta\} = \sum_{\zeta} \{\bar{\zeta}; \lambda/D\zeta\}. \tag{4.22}$$

Once more these new, remarkably simple, formulae, involving only the direct sum of positive terms, have an almost trivial interpretation in terms of tensors. This arises because the particular subgroup U(k) of O($2k$) and Sp($2k$) corresponds to the intersection of these two groups. That is, if the matrices of a set of orthogonal matrices are restricted to be symplectic, or if the matrices of a set of symplectic matrices are restricted to be orthogonal, then the resulting matrices are each decomposed, up to equivalence under similarity transformations, into a direct sum of a unitary matrix and its contragredient (Murnaghan 1958, p 111). For the elements of O($2k$) the requirement that they also belong to Sp($2k$) corresponds firstly to the application of (4.7) to give tensors which are traceless under contractions with the antisymmetric metric tensor. Then the tensor indices associated with the S-function $\{\lambda/\beta\}$ may be either covariant or contragredient leading to the branching rule (4.21). The tracelessness of the resulting tensors is an automatic consequence of tracelessness under contractions with the two distinct, symmetric and antisymmetric, metrics. The rule (4.22) follows in exactly the same way, whilst (4.20) is an immediate consequence of (4.12) and (4.21). It should be stressed that the very succinct statement of the rules is due to the adoption of the composite notation for irreducible representations of U(k).

The rules (4.5), (4.6), (4.7), (4.8), (4.12), (4.13), (4.14), (4.15), (4.20), (4.21) and (4.22) solve for covariant tensor representations all but one of the branching rule problems posed by Whippman. Each of these rules finally consists of a formula involving a direct sum of positive terms, even though negative terms appeared in intermediate stages of the calculations as a result of using the inversions (4.10) and (4.11).

5. Branching rules for mixed tensor and spinor representations

It is necessary for complete generality to discuss the mixed tensor representations $\{\bar{\mu}; \lambda\}$ of U(n) and the spinor representations $[\Delta; \lambda]$ of O(n). In addition if all associate representations are to be dealt with it is necessary to give the branching rules for the representation ϵ.

This latter problem is trivial since ϵ is the representation defined by the map from matrices to their determinants. Using well known properties of determinants the branching rules are easily seen to be

$U(n)\downarrow U(n-1)$	$\{1^n\}\downarrow\{1^{n-1}\}$	ie $\epsilon\downarrow\epsilon$	(5.1)
$U(n)\downarrow O(n)$	$\{1^n\}\downarrow\{1^n\}$	ie $\epsilon\downarrow\epsilon$	(5.2)
$U(2k)\downarrow Sp(2k)$	$\{1^{2k}\}\downarrow\langle 0\rangle$	ie $\epsilon\downarrow 1$	(5.3)
$U(p+q)\downarrow U(p)\times U(q)$	$\{1^{p+q}\}\downarrow\{1^p\}\times\{1^q\}$	ie $\epsilon\downarrow\epsilon,\epsilon$	(5.4)
$O(n)\downarrow O(n-1)$	$[1^n]\downarrow[1^{n-1}]$	ie $\epsilon\downarrow\epsilon$	(5.5)
$O(p+q)\downarrow O(p)\times O(q)$	$[1^{p+q}]\downarrow[1^p]\times[1^q]$	ie $\epsilon\downarrow\epsilon,\epsilon$	(5.6)
$O(2k+1)\downarrow U(k)$	$[1^{2k+1}]\downarrow\{0\}$	ie $\epsilon\downarrow 1$	(5.7)
$O(2k)\downarrow U(k)$	$[1^{2k}]\downarrow\{0\}$	ie $\epsilon\downarrow 1$.	(5.8)

These results may also be derived, of course, using the results of the previous section, together with the appropriate modification rules.

In principle these results allow all the branching rules given in § 4 for the covariant tensor representations, $\{\lambda\}$, of $U(n)$ to be generalized to those appropriate to the mixed tensor representations $\{\bar{\mu};\lambda\}$, by making use of (2.1). However, an alternative approach leading to branching rules for such representations, $\{\bar{\mu};\lambda\}$, of $U(n)$, valid for all values of n, may be based on the use of (3.5).

For example this relationship (3.5) implies, in combination with (4.5), the validity of the branching rule

$$U(n)\downarrow U(n-1)\qquad \{\bar{\mu};\lambda\}\downarrow\sum_{\zeta}(-1)^z\{\overline{\mu/\zeta M}\}\,.\,\{\lambda/\tilde{\zeta}M\} = \sum_{\zeta,\rho}(-1)^z\{\overline{\mu/\zeta M\rho};\lambda/\tilde{\zeta}M\rho\}$$

by virtue of (3.4). The mutual inverse nature of (3.4) and (3.5), arising from the identity (3.3), is such that the branching rule may be written in the form

$$U(n)\downarrow U(n-1)\qquad \{\bar{\mu};\lambda\}\downarrow\sum_{m,m'}\{\overline{\mu/m};\lambda/m'\} = \{\overline{\mu/M};\lambda/M\}. \tag{5.9}$$

This is the direct generalization of (4.5) which was required.

In precisely the same way if the summations over the terms m and m' of the two distinct S-function series M are replaced by summations over the terms δ and δ' of two distinct S-functions series D, and then by summations over the terms β and β' of two distinct S-function series B, the branching rules (4.6) and (4.7) may be generalized to give

$$U(n)\downarrow O(n)\qquad \{\bar{\mu};\lambda\}\downarrow\sum_{\delta,\delta'}[(\mu/\delta)\,.\,(\lambda/\delta')] = [(\mu/D)\,.\,(\lambda/D)] \tag{5.10}$$

$$U(2k)\downarrow Sp(2k)\qquad \{\bar{\mu};\lambda\}\downarrow\sum_{\beta,\beta'}\langle(\mu/\beta)\,.\,(\lambda/\beta')\rangle = \langle(\mu/B)\,.\,(\lambda/B)\rangle, \tag{5.11}$$

where use has been made of the fact that the tensor representations of $O(n)$ and $Sp(2k)$ are all self-contragredient. The brackets (. . .) have been used for punctuation purposes.

A generalization of (4.8) has been conjectured elsewhere (Abramsky 1969, Abramsky and King 1970), and its validity may be proved by using (3.5), (4.8) and (3.4) in turn to give

$$U(p+q)\downarrow U(p)\times U(q)\qquad \{\bar{\mu};\lambda\}\downarrow\sum_{\zeta,\sigma,\tau,\eta,\rho}(-1)^z\{\overline{\mu/\zeta\sigma\eta};\lambda/\tilde{\zeta}\tau\eta\}\times\{\overline{\sigma/\rho};\tau/\rho\},$$

which leads to the two equivalent forms of the branching rule:

$\mathrm{U}(p+q) \downarrow \mathrm{U}(p) \times \mathrm{U}(q)$

$$\{\bar{\mu};\lambda\} \downarrow \sum_{\sigma,\tau,\rho} \{\overline{\mu/\sigma};\lambda/\tau\} \times \{\overline{\sigma/\rho};\tau/\rho\} = \sum_{\theta,\phi,\rho} \{\overline{\mu/\rho\theta};\lambda/\rho\phi\} \times \{\bar{\theta};\phi\}. \tag{5.12}$$

This completes the derivation of the branching rules appropriate to mixed tensor representations of U(n) for the subgroups discussed in § 4.

Just as the relations (3.4) and (3.5) allowed these derivations to be made, so the relations (3.12) and (3.13) may be used to derive the branching rules appropriate to the spinor representations $[\Delta;\lambda]$ of O(n). It is first necessary to obtain the branching rules for the fundamental spin representation Δ. These are given by:

$$\mathrm{O}(2k) \downarrow \mathrm{O}(2k-1) \qquad \Delta \downarrow \Delta + \Delta^* \tag{5.13}$$

$$\mathrm{O}(2k+1) \downarrow \mathrm{O}(2k) \qquad \Delta \downarrow \Delta \tag{5.14}$$

$$\mathrm{O}(2p+2q) \downarrow \mathrm{O}(2p) \times \mathrm{O}(2q) \qquad \Delta \downarrow \Delta \times \Delta \tag{5.15}$$

$$\mathrm{O}(2p+2q+1) \downarrow \mathrm{O}(2p) \times \mathrm{O}(2q+1) \qquad \Delta \downarrow \Delta \times \Delta \tag{5.16}$$

$$\mathrm{O}(2p+2q+2) \downarrow \mathrm{O}(2p+1) \times \mathrm{O}(2q+1) \qquad \Delta \downarrow \Delta \times \Delta + \Delta^* \times \Delta^* \tag{5.17}$$

$$\mathrm{O}(2k) \downarrow \mathrm{U}(k) \qquad \Delta \downarrow \epsilon^{-1/2} \sum_{m=1}^{k} \{1^m\} = \epsilon^{-1/2} Q \tag{5.18}$$

$$\mathrm{O}(2k+1) \downarrow \mathrm{U}(k) \qquad \Delta \downarrow \epsilon^{-1/2} \sum_{m=1}^{k} \{1^m\} = \epsilon^{-1/2} Q. \tag{5.19}$$

These results may, for the most part, be established by an examination of the weights of the spin representations. There is an ambiguity in (5.17) arising from the fact that the spin representations Δ and Δ^* of both O($2p+1$) and O($2q+1$) are not completely defined. The inclusion of the factor $\epsilon^{-1/2}$ in (5.18) and (5.19), corresponding to double-valued representations of U(n), is essential in order that the corresponding matrices are self-contragredient, as are those of Δ. It is to be noted that the series appearing in (5.18) and (5.19) terminate at $m = k$, in that all the remaining terms of Q vanish identically when viewed as representations of U(k).

The derivation of branching rules appropriate to the general spinor representations $[\Delta;\lambda]$ of U(n) is then straightforward. For example the use of (3.13), (4.12), (5.13) and (3.12) yields

$$\mathrm{O}(2k) \downarrow \mathrm{O}(2k-1) \qquad [\Delta;\lambda] \downarrow (\Delta + \Delta^*) \,.\, [\lambda/PM] = [\Delta;\lambda/PMQ] + [\Delta;\lambda/PMQ]^*.$$

The S-function identity (3.7) then gives the branching rule:

$$\mathrm{O}(2k) \downarrow \mathrm{O}(2k-1) \qquad [\Delta;\lambda] \downarrow \sum_m ([\Delta;\lambda/m] + [\Delta;\lambda/m]^*) = [\Delta;\lambda/M] + [\Delta;\lambda/M]^*. \tag{5.20}$$

Similarly,

$$\mathrm{O}(2k+1) \downarrow \mathrm{O}(2k) \qquad [\Delta;\lambda] \downarrow \sum_m [\Delta;\lambda/m] = [\Delta;\lambda/M]. \tag{5.21}$$

These results are the required generalizations of (4.12). They were first derived by Boerner (1970, p 267) for the corresponding unimodular groups SO(n).

In exactly the same way,

$O(2p+2q) \downarrow O(2p) \times O(2q)$

$$[\Delta;\lambda] \downarrow (\Delta\times\Delta)\,.\sum_{\zeta}[\lambda/P\zeta]\times[\zeta/D] = \sum_{\zeta}[\Delta;\lambda/P\zeta Q]\times[\Delta;\zeta/DQ] = \sum_{\zeta}[\Delta;\lambda/\zeta]\times[\Delta;\zeta/F]$$

by virtue of (3.7) and (3.8). The branching rule then can be written in either of the two forms:

$O(2p+2q) \downarrow O(2p) \times O(2q)$

$$[\Delta;\lambda] \downarrow \sum_{\zeta,\rho}[\Delta;\lambda/\zeta]\times[\Delta;\zeta/\rho] = \sum_{\xi,\rho}[\Delta;\lambda/\rho\xi]\times[\Delta;\xi]. \tag{5.22}$$

Similarly using (5.16) and (5.17),

$$O(2p+2q+1) \downarrow O(2p) \times O(2q+1) \qquad [\Delta;\lambda] \downarrow \sum_{\zeta,\rho}[\Delta;\lambda/\zeta]\times[\Delta;\zeta/\rho] \tag{5.23}$$

and

$O(2p+2q+2) \downarrow O(2p+1) \times O(2q+1)$

$$[\Delta;\lambda] \downarrow \sum_{\zeta,\rho}([\Delta;\lambda/\zeta]\times[\Delta;\zeta/\rho]+[\Delta;\lambda/\zeta]^*\times[\Delta;\zeta/\rho]^*) \tag{5.24}$$

where the indistinguishability of $[\Delta;\mu]$ and $[\Delta;\mu]^*$ should be recalled for $O(2p+1)$ and $O(2q+1)$.

Finally using (5.18),

$O(2k) \downarrow U(k)$

$$[\Delta;\lambda] \downarrow \epsilon^{-1/2}Q\,.\sum_{\zeta}\{\bar{\zeta};\lambda/P\zeta B\} = \epsilon^{-1/2}\sum_{\zeta}\{\overline{\zeta/Q};(\lambda/P\zeta B)\,.\,Q\} = \epsilon^{-1/2}\sum_{\xi}\{\bar{\xi};(\lambda/P\xi QB)\,.\,Q\},$$

so that as a result of (3.7) the branching rule takes the form:

$$O(2k) \downarrow U(k) \qquad [\Delta;\lambda] \downarrow \epsilon^{-1/2}\sum_{\xi,\beta,m}\{\bar{\xi};(\lambda/\xi\beta)\,.\,1^m\} = \epsilon^{-1/2}\sum_{\xi}\{\bar{\xi};(\lambda/\xi B)\,.\,Q\}. \tag{5.25}$$

In the same way

$$O(2k+1) \downarrow U(k) \qquad [\Delta;\lambda] \downarrow \epsilon^{-1/2}\sum_{\xi}\{\bar{\xi};(\lambda/MP\xi QB)\,.\,Q\}$$

which leads with (3.8) to the branching rule:

$$O(2k+1) \downarrow U(k) \qquad [\Delta;\lambda] \downarrow \epsilon^{-1/2}\sum_{\xi,\zeta,m}\{\bar{\xi}:(\lambda/\xi\zeta)\,.\,1^m\} = \epsilon^{-1/2}\sum_{\xi}\{\bar{\xi}:(\lambda/\xi F)\,.\,Q\}. \tag{5.26}$$

This completes the derivation of the branching rules appropriate to spinor representations of $O(n)$ for the subgroups discussed in § 4.

Once more the remarkable fact is that, by the judicious use of infinite S-function series, all of the results obtained in this section consist of expressions involving only positive terms. The only qualifying remark that needs to be added is that, in making use of the results for specific values of n, it is necessary to use the modification rules of § 2. These may involve some negative terms. The only cases for which a natural cut-off of the infinite series does not occur, by virtue of the vanishing of certain quotients, are those covered by (5.25) and (5.26). All other formulae may be used to obtain n-independent results consisting of a finite set of terms. This unfortunate aspect of (5.25) and (5.26) cannot be avoided since it is a reflection of the k-dependence of (5.18) and (5.19).

6. Branching rules involving inner products

There remains just one branching rule discussed by Whippman (1965) which is not covered by the analysis of the previous sections. This is the rule appropriate to the restriction of $\mathrm{U}(n)$ to the subgroup $\mathrm{U}(r) \times \mathrm{U}(s)$ if $n = rs$. If the elements of $\mathrm{U}(n)$ are the $n \times n$ unitary matrices, A, then this subgroup arises from restricting A to be such that $A = E \times F$ where, in the Kronecker product, E and F are unitary matrices of dimension r and s, respectively.

This restriction corresponds to the reduction of the defining representation, $\{1\}$, of $\mathrm{U}(n)$ given by

$$U(rs) \downarrow \mathrm{U}(r) \times \mathrm{U}(s) \qquad \{1\} \downarrow \{1\} \times \{1\}. \tag{6.1}$$

The branching rule for covariant tensor representations of $\mathrm{U}(n)$ is

$$\mathrm{U}(rs) \downarrow \mathrm{U}(r) \times \mathrm{U}(s) \qquad \{\lambda\} \downarrow \sum_{\zeta} \{\lambda \circ \zeta\} \times \{\zeta\}. \tag{6.2}$$

This rule was given by Whippman and its justification lies in the fact that on restriction to the subgroup each tensor index of the representation $\{\lambda\}$ of $\mathrm{U}(rs)$ is associated with a pair of indices ranging separately over the values $1, 2, \ldots, r$ and $1, 2, \ldots, s$. It is the properties of these pairs of indices under simultaneous permutations which is relevant. The overall symmetry will be given by (λ) if the symmetries of the sets of indices are given by (η) and (ζ) provided that the inner product of S-functions $\{\eta\} \circ \{\zeta\}$ contains $\{\lambda\}$, or equivalently, provided that $\{\eta\}$ is contained in $\{\lambda\} \circ \{\zeta\}$. It is to be noted here that (λ), (η) and (ζ) are all partitions of the same number l. The inner products, $\{\eta\} \circ \{\zeta\}$ and $\{\lambda\} \circ \{\zeta\}$, define Kronecker products of irreducible representations of S_l.

Although formally (6.2) is very similar to (4.8) an important distinction needs to be made between these two results. To make use of them it is necessary to evaluate the coefficients $m^{\lambda}_{\zeta\eta}$ and $k_{\lambda\zeta\eta}$ defined by (3.1) and (3.6). In both cases there exist copious tabulations of these coefficients in the literature (Itzykson and Nauenberg 1966, Wybourne 1970, Vanagas 1971). However the algorithm based on the Littlewood–Richardson rule (Littlewood 1950a, p 94) leading to the evaluation of $m^{\lambda}_{\zeta\eta}$ gives the answer directly as a sum of positive terms, whilst all algorithms leading to the evaluation of $k_{\lambda\zeta\eta}$ involve negative terms at intermediate stages.

The method given by Robinson (1961, p 64), for example, corresponds to the use of the formula

$$k_{\lambda\zeta\eta} = \sum_{r \geqslant s \geqslant \ldots \geqslant t > 0} \sum_{\rho, \sigma, \ldots, \tau} b^{rs \ldots t}_{\lambda} m^{\zeta}_{\rho\sigma\ldots\tau} m^{\eta}_{\rho\sigma\ldots\tau}, \tag{6.3}$$

where

$$\{\rho\} . \{\sigma\} . \ldots . \{\tau\} = \sum_{\xi} m^{\xi}_{\rho\sigma\ldots\tau} \{\xi\}, \tag{6.4}$$

and the special case of this formula, involving the outer product of symmetric S-functions,

$$\{r\} . \{s\} . \ldots . \{t\} = \sum_{\lambda} m^{\lambda}_{rs\ldots t} \{\lambda\}, \tag{6.5}$$

has as its inverse:

$$\{\lambda\} = \sum_{r \geq s \geq \ldots \geq t > 0} b_\lambda^{rs\ldots t}\{r\} . \{s\} . \ldots . \{t\}. \tag{6.6}$$

This remarkable formula (6.3) follows from the special case of (6.2):

$$\mathrm{U}(rs) \downarrow \mathrm{U}(r) \times \mathrm{U}(s) \qquad \{t\} \downarrow \sum_\tau \{\tau\} \times \{\tau\}, \tag{6.7}$$

and the use of (6.6) and (6.4).

Even though it is well known (Robinson 1961, p 42, Blaha 1969, King and Plunkett 1972) that it is not necessary to invert (6.5) explicitly to obtain the coefficients $b_\lambda^{rs\ldots t}$ appearing in (6.6), the coefficients necessarily take on negative as well as positive values. Thus the use of (6.3) involves a certain amount of overcounting of terms which eventually cancel. This defect apart, it is interesting to note that (6.3) gives a relationship, albeit complicated, between inner and outer products, and of course (6.3) provides a method of evaluating the branching rule (6.2) as required.

This branching rule may be extended to cover the case of mixed tensor representations in the two ways suggested in § 5. The first way is to make use of the branching rule appropriate to the representation ϵ in which A is mapped onto det A. This takes the form:

$$\mathrm{U}(rs) \downarrow \mathrm{U}(r) \times \mathrm{U}(s) \qquad \{1^{rs}\} \downarrow \{s^r\} \times \{r^s\} \qquad \text{ie } \epsilon \downarrow \epsilon^s \times \epsilon^r \tag{6.8}$$

as can be seen either by using (6.2) and the appropriate modification rules, or by noting (Murnaghan 1938, p 70) that

$$\det(E \times F) = (\det E)^s \times (\det F)^r \tag{6.9}$$

if E and F are $r \times r$ and $s \times s$ matrices, respectively.

This result (6.8), together with (6.2), allows the branching rules appropriate to the mixed tensor representations $\{\bar{\mu}; \lambda\}$ of U(rs) to be written down making use of (2.1). A separate calculation has to be carried out for each distinct pair of values for r and s, and, for all but small values of $n = rs$, (ρ) will be a partition of such a large number that the evaluation of the branching rule (6.2) appropriate to $\{\rho\}$ using (6.3) will be extremely tedious.

This difficulty may be avoided by using the second way based on the identities (3.4) and (3.5). These yield the general result (King 1970a):

$$\mathrm{U}(rs) \downarrow \mathrm{U}(r) \times \mathrm{U}(s) \qquad \{\bar{\mu}; \lambda\} \downarrow \sum_{\zeta, \sigma, \tau, \eta, \rho} (-1)^z \{\overline{((\mu/\zeta) \circ \sigma)/\eta}; (\lambda/\tilde{\zeta}) \circ \tau\} \times \{\overline{\sigma/\rho}; \tau/\rho\}. \tag{6.10}$$

Unfortunately due to the mixing in this formula of inner products and quotients, rather than the mutually inverse quantities: outer products and quotients, no obvious simplification occurs. In particular it is not possible to demonstrate in general the cancellation of all negative terms in (6.10) to give a series of positive terms.

In addition to the branching rules discussed by Whippman there are three further rules related to (6.2) which he did not mention. These are the rules appropriate to the embedding of O(r) × O(s) in O(rs), Sp($2r$) × Sp($2s$) in O($4rs$) and Sp($2r$) × O(s) in Sp($2rs$). Exactly as in § 4 a consideration of subgroup chains such as

$$\mathrm{O}(rs) \uparrow_r \mathrm{U}(rs) \downarrow \mathrm{U}(r) \times \mathrm{U}(s) \downarrow \mathrm{O}(r) \times \mathrm{O}(s)$$

yields, through the use of (6.2), the results:

$$\mathrm{O}(rs) \downarrow \mathrm{O}(r) \times \mathrm{O}(s) \qquad [\lambda] \downarrow \sum_{\zeta} [((\lambda/C) \circ \zeta)/D] \times [\zeta/D], \tag{6.11}$$

$$\mathrm{O}(4rs) \downarrow \mathrm{Sp}(2r) \times \mathrm{Sp}(2s) \qquad [\lambda] \downarrow \sum_{\zeta} \langle ((\lambda/C) \circ \zeta)/B \rangle \times \langle \zeta/B \rangle, \tag{6.12}$$

$$\mathrm{Sp}(2rs) \downarrow \mathrm{Sp}(2r) \times \mathrm{O}(s) \qquad \langle \lambda \rangle \downarrow \sum_{\zeta} \langle ((\lambda/A) \circ \zeta)/B \rangle \times [\zeta/D]$$

$$= \sum_{\zeta} \langle \zeta/B \rangle \times [((\lambda/A) \circ \zeta)/D]. \tag{6.13}$$

Just as in (6.10) these expressions involve both positive and negative terms, as can be seen from the definitions of the infinite S-function series A and C given in § 3. Nevertheless for tensor representations they provide a complete answer to the branching rule problems for these particular embeddings.

For the associates of such tensor representations it is sometimes convenient to make use of the branching rules:

$$\mathrm{O}(rs) \downarrow \mathrm{O}(r) \times \mathrm{O}(s) \qquad [1^{rs}] \downarrow [1^r]^s \times [1^s]^r \qquad \text{ie } \epsilon \downarrow \epsilon^s \times \epsilon^r \tag{6.14}$$

$$\mathrm{O}(4rs) \downarrow \mathrm{Sp}(2r) \times \mathrm{Sp}(2s) \qquad [1^{4rs}] \downarrow \langle 0 \rangle \times \langle 0 \rangle \qquad \text{ie } \epsilon \downarrow 1 \times 1 \tag{6.15}$$

as follows from (6.9). Of course in (6.14), $\epsilon^s = \epsilon^{s(\mathrm{mod}\ 2)}$ and $\epsilon^r = \epsilon^{r(\mathrm{mod}\ 2)}$. In contrast to this it is not possible to deal with the spinor representations of $\mathrm{O}(n)$ so easily. A consideration of the weights of the basic spin representations suggests the validity of the formulae:

$$\mathrm{O}(4rs) \downarrow \mathrm{O}(2r) \times \mathrm{O}(2s) \qquad \Delta \downarrow \sum_{\zeta} [s^r/\zeta] \times [\tilde{\zeta}] \tag{6.16}$$

$$\mathrm{O}(4rs+2s) \downarrow \mathrm{O}(2r+1) \times \mathrm{O}(2s) \qquad \Delta \downarrow \sum_{\zeta} [s^r/\zeta] \times [\Delta; \tilde{\zeta}] \tag{6.17}$$

$$\mathrm{O}(4rs+2r+2s+1) \downarrow \mathrm{O}(2r+1) \times \mathrm{O}(2s+1) \qquad \Delta \downarrow \sum_{\zeta} [\Delta; s^r/\zeta] \times [\Delta; \tilde{\zeta}] \tag{6.18}$$

$$\mathrm{O}(4rs) \downarrow \mathrm{Sp}(2r) \times \mathrm{Sp}(2s) \qquad \Delta \downarrow \sum_{\zeta} \langle s^r/\zeta \rangle \times \langle \tilde{\zeta} \rangle. \tag{6.19}$$

The ambiguity in the distinction between spinor representations of $\mathrm{O}(2k+1)$ and their associates is carried over into these expressions. Moreover, although the analysis of the corresponding projections in weight space indicates that all the terms given must appear, this analysis does not immediately preclude the presence of other terms. Applications to specific examples of fairly low dimension indicate that no terms other than those given in these formulae are present however.

These results may then be used in combination with the identities (3.12) and (3.13) and the branching rules (6.11), (6.12) and (6.13) to give the results appropriate to composite spinor representations $[\Delta; \lambda]$. For example,

$$\mathrm{O}(4rs) \downarrow \mathrm{O}(2r) \times \mathrm{O}(2s) \qquad [\Delta; \lambda] \downarrow \sum_{\xi} [s^r/\xi] \times [\tilde{\xi}] \,.\, \sum_{\zeta} [((\lambda/PC) \circ \zeta)/D] \times [\zeta/D]$$

$$= \sum_{\xi, \zeta, \eta, \rho} [(s^r/\xi\eta) \,.\, ((\lambda/E) \circ \zeta)/D\eta] \times [(\tilde{\xi}/\rho) \,.\, (\zeta/D\rho)], \tag{6.20}$$

and similarly in other cases. No obvious simplifications occur in this formula or in others derived in the same way.

7. Conclusion

It is clear that the results given in the previous sections separate into two distinct categories. Firstly, for the eleven restrictions $\mathrm{U}(n) \downarrow \mathrm{U}(n-1)$, $\mathrm{U}(n) \downarrow \mathrm{O}(n)$, $\mathrm{U}(2k) \downarrow \mathrm{Sp}(2k)$, $\mathrm{U}(p+q) \downarrow \mathrm{U}(p) \times \mathrm{U}(q)$, $\mathrm{O}(n) \downarrow \mathrm{O}(n-1)$, $\mathrm{Sp}(2k) \downarrow \mathrm{Sp}(2k-2)$, $\mathrm{O}(p+q) \downarrow \mathrm{O}(p) \times \mathrm{O}(q)$, $\mathrm{Sp}(2p+2q) \downarrow \mathrm{Sp}(2p) \times \mathrm{Sp}(2q)$, $\mathrm{O}(2k+1) \downarrow \mathrm{U}(k)$, $\mathrm{O}(2k) \downarrow \mathrm{U}(k)$ and $\mathrm{Sp}(2k) \downarrow \mathrm{U}(k)$, general formulae have been obtained covering all types of representation: covariant tensor, mixed tensor and spinor. These formulae involve only the addition, outer multiplication and division of S-functions. They are valid for arbitrary values of n, k, p and q.

Secondly, for the four restrictions $\mathrm{U}(rs) \downarrow \mathrm{U}(r) \times \mathrm{U}(s)$, $\mathrm{O}(rs) \downarrow \mathrm{O}(r) \times \mathrm{O}(s)$, $\mathrm{O}(4rs) \downarrow \mathrm{Sp}(2r) \times \mathrm{Sp}(2s)$ and $\mathrm{Sp}(2rs) \downarrow \mathrm{Sp}(2r) \times \mathrm{O}(s)$, general formulae have been derived for all covariant tensor and mixed tensor representations and implied for all spinor representations. However these formulae now involve both the subtraction and the inner multiplication of S-functions. As such they are more difficult to apply in practice. They are valid for arbitrary values of r and s.

Clearly these branching rules do not exhaust all the rules appropriate to the embedding of one classical Lie group in another. Indeed the unimodular restrictions $\mathrm{U}(n) \downarrow \mathrm{SU}(n)$ and $\mathrm{O}(n) \downarrow \mathrm{SO}(n)$, whose branching rules are well known, have not been discussed at all, nor have restrictions of the type $\mathrm{U}(n) \times \mathrm{U}(n) \downarrow \mathrm{U}(n)$, whose branching rules involve Kronecker products of representations, nor have restrictions of the type $\mathrm{U}(n) \downarrow \mathrm{U}(m)$ whose branching rules, as emphasized by Wybourne (1970, p 64), involve plethysms: the new multiplication of S-functions introduced by Littlewood (1950a, p 289): nor have restrictions of the unimodular groups such as $\mathrm{SO}(p+q) \downarrow \mathrm{SO}(p) \times \mathrm{SO}(q)$, for which the branching rules of § 4 and § 5 are insufficient to deal with all irreducible representations.

It is not appropriate to consider all these here. Suffice it to say that S-function techniques are adequate to deal with them, save in the case of certain irreducible representations of $\mathrm{SO}(2k)$ which can best be dealt with using the method of difference characters (Littlewood 1950a, pp 246, 259, Butler and Wybourne 1969) or related weight space techniques.

Such weight space techniques provide a means of calculating all branching rules of the unimodular classical and exceptional Lie groups as emphasized, for example, by Navon and Patera (1967), Stone (1970), Wong (1970) and Gruber (1970). This is best demonstrated in the work of Patera and Sankoff (1972) who tabulated the branching rules for all irreducible representations of dimension less than 1000 appropriate to each of the maximal embeddings of a semi-simple Lie algebra in a simple Lie algebra of rank less than nine. This work is complementary to that given here in as much as their results are appropriate to all unimodular Lie groups, both classical and exceptional, treated one by one in order to obtain specific results for low rank algebras and low dimensional representations. In the present paper the emphasis has been placed on the non-unimodular classical Lie groups, and general results applicable to arbitrary rank algebras and representations of arbitrarily high dimension have been obtained.

No specific results have been given here and not all embeddings have been discussed. Nevertheless, of the 142 maximal embeddings of classical Lie groups in classical Lie groups listed by Patera and Sankoff, 122 are covered by the branching rules of §§ 4, 5 and 6, together with unimodular restrictions which are trivial in all but the cases of a few representations. The branching rules appropriate to the remaining 20 embeddings all involve plethysms.

One great advantage of the use of the general formulae obtained here as compared with the weight space techniques is that the overcounting necessarily associated with analysing projections between weight spaces is avoided completely in the case of branching rules involving only the addition, outer multiplication and division of S-functions. It is greatly reduced in those branching rules involving, in addition, only the subtraction and inner products of S-functions. Even in the case of those involving plethysms it may be reduced by a variety of methods. This is because plethysms may themselves be calculated not only via projections between weight spaces (King and Plunkett 1972), but also by many other methods, some of which are recursive (Littlewood 1950a, p 290, Wybourne 1970, p 52, Butler 1970, Plunkett 1972, Butler and King 1973). The adoption of these methods has led to a number of important tabulations of plethysms directly related to branching rules by Ibrahim (1970), Butler and Wybourne (1971) and Vanagas (1971).

The fact that Patera and Sankoff (1972) limit their calculations to low rank algebras is due just to limitations of time and space in their computer calculations. No fundamental limit exists. However their method involving weight space techniques cannot lead to any general results valid for arbitrary rank algebras. This is because a separate calculation is necessary for each distinct value of the rank of an algebra. This in turn is due, in part, to the fact that the multiplicities of weights of representations are rank dependent in the cases of B_k, C_k and D_k, corresponding to the groups SO$(2k+1)$, Sp$(2k)$ and SO$(2k)$. This is not so in the case of A_k for which the corresponding group is SU$(k+1)$. The multiplicities are given (Delaney and Gruber 1969, Blaha 1969, King and Plunkett 1972) by the coefficients $m^{\lambda}_{rs\ldots t}$ in the k-independent formula (6.5). No similar formula exists for the other classical groups. However the branching rules derived here in §§ 4 and 5 may be used to determine such multiplicities. Considerable progress has been made in this direction by Plunkett (1971) using a method based on the derivation of the branching rules (4.20), (4.21), (4.22), (5.25) and (5.26) appropriate to O$(2k+1)\downarrow$ U(k), O$(2k)\downarrow$ U(k) and Sp$(2k)\downarrow$ U(k).

The really striking feature of the results obtained here is that the S-function techniques are such that many of the final formulae have a very simple interpretation. This indeed is true of all the results given in § 4. In particular the remark of Wong (1970), "It is interesting (though quite mysterious!) to note that all these branching rules look very similar to each other", made in relation to the restrictions U$(n)\downarrow$ U$(n-1)$, O$(n)\downarrow$ O$(n-1)$ and Sp$(2k)\downarrow$ Sp$(2k-2)$ should now be modified in the sense that his parenthetical exclamation is now unnecessary.

A final reiteration of the warning given in the introduction regarding possible confusion between different embeddings of one group in another is in order. Like all the embeddings discussed by Patera and Sankoff most of those analysed here are maximal. Amongst them the rules appropriate to O$(2k)\downarrow$ U(k) and Sp$(2k)\downarrow$ U(k) are the generalization, aimed at by Whippman (1965), of the results he established for low rank cases through the local isomorphisms of certain groups. Clearly the branching rule for O(8) $\downarrow$ U(4) given by (4.21) and (5.25) is quite distinct from that appropriate to the chain O(8) $\downarrow$ O(7) $\downarrow$ O(6) even though SU(4) and SO(6) are locally isomorphic. Similarly if comparison is made with the work of Wong (1970) it is essential to distinguish between the restrictions O(9) $\downarrow$ O(3) $\times$ O(3), with the branching rule given by (6.11), and the chain O(9) $\downarrow$ O(6) $\times$ O(3) $\downarrow$ O(5) $\times$ O(3) $\downarrow$ O(4) $\times$ O(3) $\downarrow$ O(3) $\times$ O(3) for which the branching rule is given by (4.14) and the successive use of (4.12) three times.

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